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Electromagnetic waves in ferrites: from linear absorption to the nonlinear Schrödinger equation

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Abstract. We examine the effect of damping on the nonlinear modulation of an electromagnetic plane wave in a ferrite. Depending on the value of the damping constant, the time evolution of the amplitude of the wave is either a simple exponential decay, or is described either by a nonlinear Schrödinger (NLS) equation, or by a perturbed NLS equation. We give a new exact solution to this latter equation, and a way to compute approximate solutions.

1. Introduction

The problem of wave propagation in ferromagnetic dielectrics has been studied by many authors, mainly from a linear point of view [1]. The interest in this matter is especially due to its applications to ferrite devices at microwave frequencies [2, 3], but the problem is in fact highly nonlinear. The development of the method of multiscale expansions, and of soliton theory, has lead to renewed interest.

The present author, in collaboration with Manna, published some time ago a study of the nonlinear modulation of an electromagnetic monochromatic plane wave in such a medium [4]. It was shown that it obeys the nonlinear Schrödinger (NLS) equation. A detailed study of the coefficients of this equation led to the characterization of focusing and defocusing regimes for the wave [5]. This effect has also been observed experimentally [6–8].

In our previous works damping was neglected. Now we could like to take it into account in the following way. Using the same model as in [4, 5], we introduce into the basic equations a phenomenological term that describes the damping. Then we apply the same multiscale method, and see how the damping affects the modulation of the wave. This way, we follow the example of Nakata. Dealing with the same model, he described in a first paper [9] a mode of solitonic waves that can propagate in a ferromagnetic medium, and showed that the time evolution of this mode is governed by the modified Korteweg–de Vries (mKdV) equation. Later he published a second study giving the equation that describes the effect of the damping on such a wave [10]. Notice that Nakata's work concerns solitary waves, which may be considered as a limiting case of waves with a very small wave number and frequency, whereas we are dealing here with a problem of wave modulation, which is rather a question of fast oscillating waves. Nakata also obtained an asymptotic model governed by the so-called derivative NLS equation [11]. Despite the equations obtained being close together, they have totally different meanings. In Nakata's work, the complex field represents the two-dimensional component of the magnetic field (or of the

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magnetization density) in the plane transverse to the propagation direction, whereas in our case it is the complex amplitude of a fast oscillating wave.

The result obtained, in Nakata's paper as well as in the present one, depends strongly on the order of magnitude of the damping constant σ. In our case, if σ is small enough, the NLS asymptotic describes the modulation, at the amplitude, space and time scales under consideration. If σ is large, the damping term induces an exponential decay of the amplitude of the wave that hides any nonlinear behaviour. Between these two extreme cases there is an order of magnitude of σ for which the damping balances the nonlinearity. An equation describing this case is derived: it is a perturbed NLS equation. This equation has already been given in the same physical context [7, 12], and also in nonlinear optics [13] and hydrodynamics [14]. It has the following form:

$$\mathrm{i}\frac{\partial g}{\partial t} + B\frac{\partial^2 g}{\partial x^2} + Cg|g|^2 = \mathrm{i}Dg \tag{1}$$

(g is here the complex amplitude of the wave). Such an equation has completely different properties depending on whether the coefficients are real or not. If both B and C have a non-zero imaginary part, it is the Ginzburg–Landau (GL) equation that has explicit [15–17], but also chaotic, solutions [18, 19]. If B and C are real, it is a perturbed NLS equation and does not have any of the properties of the GL equation.

The derivation used in [12] is rather heuristic and fails in computing the coefficients. Our first contribution in this paper is to compute them explicitly and prove that the equation obtained is actually the NLS perturbed one, and not the GL equation.

Second, we discuss precisely which of the three mentioned behaviours are obtained, depending on the order of magnitude of the damping constant σ. This is very important, because it justifies the following experimental fact observed by De Gasperis *et al* [6]: above some threshold in the power input, the absorption falls. This is due to the fact that the unperturbed NLS equation must replace the other models over some power threshold.

Third, we recall the known properties of the perturbed NLS equation and give a way to compute one exact and several approximate solutions for it. This is done in the second part of the paper.

The question of whether the inhomogeneous exchange term could be neglected or not had to be analysed in this paper. The result is that this term does not modify the main nonlinear behaviour of the wave, a fact which is proved in appendix A.

2. The model and the multiscale expansion

We use, as in [4, 5], a classical model based on the Maxwell equations, and on the equation that governs the evolution of a magnetic moment in a magnetic field. This is the model commonly used to study the behaviour of electromagnetic waves in ferrites [1, 3, 20], mainly at microwave frequencies. In particular, it is used in the theory of ferromagnetic resonance [2, 21]. This model is macroscopic and neglects the effect of anisotropy, the inhomogeneous exchange interaction, domain walls, and the finite size of the sample. A suitable choice of material will allow us to neglect anisotropy. We assume that the sample is immersed in a constant exterior field $\boldsymbol{H}_{\mathrm{ext}}$, as is usual in ferromagnetic resonance experiments. The field $\boldsymbol{H}_{\mathrm{ext}}$ should be strong enough to magnetize the sample to saturation. This allows us to neglect the existence of domain walls and the effects of finite size of the sample.

Is the inhomogeneous exchange interaction to be taken into account? It is *a priori* negligible because we are considering here microwave frequencies, and not spin waves. The inhomogeneous exchange term is important for the study of wave propagation in thin films [22], but the geometry of the present problem is totally different. However, interaction with spin waves can occur and be responsible for nonlinear damping of the microwave frequency. This effect has been theoretically studied by Suhl in the frame of the ferromagnetic resonance [23]. Our model does not take into account these interactions which can be avoided experimentally [6].

However, when we consider the effect of a weak damping term, the question of whether the inhomogeneous exchange interaction term is still negligible can arise. We are able to give here a precise answer: taking into account this term modifies the asymptotic model only by a linear phase factor. This factor gives the first correction to the linear dispersion relation due to the inhomogeneous exchange interaction. The coefficients of the nonlinear equation are not modified at all. The justification of this fact is lengthy, but seems to us to be useful: see appendix A.

If we assume a linear relationship $\boldsymbol{D} = \tilde{\varepsilon}\boldsymbol{E}$ between the electric field $\boldsymbol{E}$ and the electric induction $\boldsymbol{D}$, the Maxwell equations reduce to

$$-\nabla(\nabla\cdot\boldsymbol{H}) + \Delta\boldsymbol{H} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}(\boldsymbol{H}+\boldsymbol{M}) \tag{2}$$

where $\boldsymbol{H}$, $\boldsymbol{M}$, and $c = 1/\sqrt{\mu_0\tilde{\varepsilon}}$ are, the magnetic field, the magnetization density, and the speed of light based on the dielectric constant $\tilde{\varepsilon}$, respectively. The evolution of the magnetization density $\boldsymbol{M}$ is governed by the following equation,

$$\frac{\partial\boldsymbol{M}}{\partial\boldsymbol{t}} = -\delta\mu_0\boldsymbol{M}\wedge\boldsymbol{H} + \frac{\sigma}{\|\boldsymbol{M}\|}\boldsymbol{M}\wedge(\boldsymbol{M}\wedge\boldsymbol{H}) \tag{3}$$

where δ is the gyromagnetic ratio and σ a negative constant. Neglecting the second term proportional to σ, we have the torque equation, describing the evolution of a magnetic moment $\boldsymbol{M}$ in a magnetic field $\boldsymbol{H}$. This equation, always valid at the microscopic scale, is also valid at a macroscopic scale in a ferromagnet below the Curie point. We used it in [4, 5]. The term $(\sigma/\|\boldsymbol{M}\|)\boldsymbol{M}\wedge(\boldsymbol{M}\wedge\boldsymbol{H})$ is a phenomenological term that describes the damping: for a free system $\boldsymbol{M}$ tends to line up with $\boldsymbol{H}$, instead of having a constant precession motion around this vector. It was first proposed by Landau and Lifchitz [20], and several forms have been given for it by various authors [3, 24]. Notice that in regard to the perturbative calculus of the present paper, all these forms are equivalent (especially, equation (3) shows that $(\partial/\partial t)\|\boldsymbol{M}\|^2 = 0$, and therefore the whole term $\sigma/\|\boldsymbol{M}\|$ is a simple constant).

The constant σ can be measured by means of the line width in a standard ferromagnetic resonance absorption experiment. We have [3 (p 73), 24, 25]

$$|\sigma| = \mu_0\delta\frac{\Delta H}{2H_0} \tag{4}$$

where H_0 is the resonance exterior field and ΔH the linewidth of the resonance curve. Many experimental data can be found in published studies. The dimensionless parameter

$$\tilde{\sigma} = \frac{\sigma}{\mu_0\delta} \tag{5}$$

always has a low value that allows us to treat the damping term as a perturbation. In fact, σ is not a constant, and depends on the frequency ω of the wave, but the effects of this dependency are of higher order and we can neglect it.

After rescaling $\boldsymbol{M}$, $\boldsymbol{H}$, t into $\delta\mu_0\boldsymbol{M}/c$, $\delta\mu_0\boldsymbol{H}/c$, ct, equations (2) and (3) become

$$-\nabla(\nabla\cdot\boldsymbol{H}) + \Delta\boldsymbol{H} = \frac{\partial^2}{\partial t^2}(\boldsymbol{H}+\boldsymbol{M}) \tag{6}$$

$$\frac{\partial \boldsymbol{M}}{\partial \boldsymbol{t}} = -\boldsymbol{M} \wedge \boldsymbol{H} + \frac{\tilde{\sigma}}{\|\boldsymbol{M}\|} \boldsymbol{M} \wedge (\boldsymbol{M} \wedge \boldsymbol{H}) \tag{7}$$

where $\tilde{\sigma}$ is related to the constant σ by equation (5).

We expand the fields $\boldsymbol{H}$, $\boldsymbol{M}$ in a Fourier series

$$\boldsymbol{H} = \sum_{n \in \mathbb{Z}} \boldsymbol{H}^n \mathrm{e}^{\mathrm{i}n\varphi} \tag{8}$$

$$\boldsymbol{M} = \sum_{n \in \mathbb{Z}} \boldsymbol{M}^n \mathrm{e}^{\mathrm{i}n\varphi} \tag{9}$$

where the phase φ is defined by $\varphi = kx - \omega t$, x being the spatial coordinate along the propagation direction. We have the reality condition $\boldsymbol{H}^{-n} = \boldsymbol{H}^{n*}$, $\boldsymbol{M}^{-n} = \boldsymbol{M}^{n*}$, where $*$ denotes complex conjugation. We assume that the amplitudes $\boldsymbol{H}^n$, $\boldsymbol{M}^n$ vary slowly in time and space, and neglect their transverse variation. Thus we introduce a small parameter ε, and three slow variables:

$$\begin{cases} \xi = \varepsilon x \\ T = \varepsilon t \\ \tau = \varepsilon^2 t. \end{cases} \tag{10}$$

We expand the amplitudes $\boldsymbol{H}^n$, $\boldsymbol{M}^n$ in a power series of ε:

$$\boldsymbol{H}^n = \sum_{j=0}^{\infty} \varepsilon^j \boldsymbol{H}_j^n \tag{11}$$

$$\boldsymbol{M}^n = \sum_{j=0}^{\infty} \varepsilon^j \boldsymbol{M}_j^n. \tag{12}$$

$\boldsymbol{H}_j^n$ and $\boldsymbol{M}_j^n$ are functions of ξ and τ. $\boldsymbol{H}_0^0$ and $\boldsymbol{M}_0^0$ are assumed to be constant and represent the exterior field where the sample is immersed, and the magnetization density at saturation, lined up with the field. Notice that $\boldsymbol{H}_0^0$ is not the exterior field $\boldsymbol{H}_{\text{ext}}$ itself, but the field created by $\boldsymbol{H}_{\text{ext}}$ inside the medium. It depends on $\boldsymbol{H}_{\text{ext}}$ through demagnetizing factors. Because we do not want to fix the shape of the sample, assumed to be infinite in our model, we do not write down these demagnetizing factors, and consider only $\boldsymbol{H}_0^0$, which we will call the external field, despite the fact that it differs from $\boldsymbol{H}_{\text{ext}}$.

$\boldsymbol{H}_1^1$ and $\boldsymbol{M}_1^1$ represent the slowly-varying complex amplitude of the monochromatic plane wave under consideration. All other terms (except the conjugates of the latter) of order 0 or 1 are set to zero, and we assume that the higher-order terms vanish as $\xi \longrightarrow -\infty$. This is the usual multiscale expansion that ordinarily leads to the NLS equation [26].

Considering the damping term as a perturbation, we will assume (as in [10]) that $\tilde{\sigma}$ satisfies

$$\tilde{\sigma} = \varepsilon^p \hat{\sigma} \tag{13}$$

where p is some positive integer, and $\hat{\sigma}$ a quantity of order 1. Then we put expansions (8)–(12) into the basic equations (6) and (7) and collect the terms order by order in ε.

The result will depend on the order of magnitude of the damping coefficient $\tilde{\sigma}$, that is, of the integer p. At order ε^0, we obtain, whatever the value of p,

$$\boldsymbol{H}_0 = \alpha \boldsymbol{m} \tag{14}$$

$$\boldsymbol{M}_0 = \boldsymbol{m} \tag{15}$$

where

$$\boldsymbol{m} = \begin{pmatrix} m_x \\ m_t \\ 0 \end{pmatrix}. \tag{16}$$

$\boldsymbol{m}$ is a constant vector and α a real constant.

At order ε^1, we obtain the dispersion relation

$$\mu^2 m_x^2 + \gamma\mu(1+\alpha)m_t^2 = \gamma^2\omega^2 \tag{17}$$

where

$$\mu = 1 + \alpha\gamma \tag{18}$$

$$\gamma = 1 - \frac{k^2}{\omega^2} \tag{19}$$

and

$$\boldsymbol{M}_1^1 = \boldsymbol{m}_1^1 g(\xi, T, \tau) \tag{20}$$

$$\boldsymbol{m}_1^1 = \begin{pmatrix} -\mathrm{i}\gamma\mu m_t \\ \mathrm{i}\gamma\mu m_x \\ -\gamma^2\omega \end{pmatrix} \tag{21}$$

$$\boldsymbol{H}_1^1 = \boldsymbol{h}_1^1 g(\xi, T, \tau) \tag{22}$$

$$\boldsymbol{h}_1^1 = \begin{pmatrix} \mathrm{i}\gamma\mu m_t \\ -\mathrm{i}\mu m_x \\ \gamma\omega \end{pmatrix}. \tag{23}$$

At order ε^2, the third harmonics and those following are zero; the second harmonics are non-zero and uniquely determined. To avoid unhelpful length in this article, we shall not record these values here which, as with other details of this calculus, are the same as those in case $\tilde{\sigma} = 0$, so can be omitted without inconvenience for the clarity of this paper. The reader interested in these details should see [4, 5]. In the case where $\tilde{\sigma}$ is of order ε ($p = 1$), the solvability condition for the fundamental term leads to the following equation:

$$\frac{\partial g}{\partial T} + V\frac{\partial g}{\partial \xi} + \frac{g}{T_0} = 0. \tag{24}$$

V is the group velocity of the wave,

$$V = \frac{(b+1)u}{b + 1 + \gamma\mu u^2} \tag{25}$$

with

$$u = \frac{\omega}{k} \tag{26}$$

and

$$b = \frac{\mu^2 m_x^2}{\gamma^2\omega^2}. \tag{27}$$

The constant T_0 has the following expression:

$$T_0 = \frac{m}{-\hat{\sigma}} \frac{2(1-\gamma)(1+\alpha)(b+1+\gamma\mu u^2)}{\gamma\omega^2[2\mu - (1+b)(1-\gamma)]}. \tag{28}$$

The solution of equation (24) is

$$g(\xi, T, \tau) = q(\xi - VT, \tau)\, \mathrm{e}^{-T/T_0}. \tag{29}$$

Thus the dominant time evolution of g is an exponential decay. The calculus can be pursued in order to find the τ-dependency of q (or g). We solve the equations in an analogous way to that in [4], with additional technical complications. We obtain an equation of the form

$$\mathrm{i}A\frac{\partial q}{\partial \tau} + B\frac{\partial^2 q}{\partial x^2} + E\frac{\partial q}{\partial x} + Fq + Cq|q|^2\,\mathrm{e}^{-2t/T_0} + \lambda q\varphi\,\mathrm{e}^{-2t/T_0} + I\frac{\partial r}{\partial t} = 0 \tag{30}$$

A, B, C, E, F, I, λ are constants. We have made the change of variables

$$\begin{cases} x = \xi - VT \\ t = T \\ \tau = \tau. \end{cases} \tag{31}$$

$r = r(x, t, \tau)$ and $\varphi = \varphi(x, \tau)$ are additional degrees of freedom that take place in the expressions for the quantities $\boldsymbol{H}_2^1$ and $\boldsymbol{H}_2^0$, respectively. Let

$$K = \mathrm{i}A\frac{\partial q}{\partial \tau} + B\frac{\partial^2 q}{\partial x^2} + E\frac{\partial q}{\partial x} + Fq \tag{32}$$

and

$$Q = (Cq|q|^2 + \lambda\varphi)q. \tag{33}$$

K and Q do not depend on t, thus

$$r(x, t, \tau) = \frac{-K}{I}t + \frac{QT_0}{2I}\mathrm{e}^{2t/T_0} + r_0(x, \tau) \tag{34}$$

r_0 being an arbitrary function of x, τ only. $\boldsymbol{H}_2^1$, and thus r, must be bounded as $t \longrightarrow +\infty$. Thus we have necessarily

$$K = Q = 0. \tag{35}$$

In particular, we find that

$$\mathrm{i}A\frac{\partial q}{\partial \tau} + B\frac{\partial^2 q}{\partial x^2} + E\frac{\partial q}{\partial x} + Fq = 0. \tag{36}$$

A and B have the same expression as in the NLS equations of [4, 5], and E, F are real constants, given by the formulae listed in appendix B.

The solutions of equation (36) have the form

$$q(x, \tau) = q_0 \exp\left[\mathrm{i}lx + \mathrm{i}\frac{F - Bl^2}{A}\tau - \frac{El}{A}\tau\right] \tag{37}$$

where q_0 is an arbitrary complex number, and l real. We obtain corrections to the time evolution of the wave, for the oscillatory part as well as for the exponential decay, but these corrections are still linear.

It is clear, from this calculus, that the presence of an exponential decay in the transport equation (24) kills any nonlinear effect, even at a higher order. The main interest of equation (36) lies in this last remark, even if we are not quite sure of the physical relevance of the correcting terms in (37).

In the case where $p > 1$, the term $(1/T_0)g$ in equation (24) is of higher order, and

$$g(\xi, T, \tau) = g(\xi - VT, \tau). \tag{38}$$

We find exactly the NLS frame again. The computation of terms of order ϵ^2 for $n = 0$ is described in [4, 5]. The equation describing the τ-evolution of g is the solvability condition for the fundamental harmonic $n = 1$ at order ε^3. For $p = 2$, we obtain

$$\begin{aligned} \boldsymbol{m} \wedge (\boldsymbol{H}_3^1 - \alpha \boldsymbol{M}_3^1) - \mathrm{i}\omega \boldsymbol{M}_3^1 = V \frac{\partial}{\partial \xi} \boldsymbol{M}_2^1 - \frac{\partial}{\partial \tau} \boldsymbol{M}_1^1 \\ - \sum_{r+s=1} (\boldsymbol{M}_1^r \wedge \boldsymbol{H}_2^s + \boldsymbol{M}_2^r \wedge \boldsymbol{H}_1^s) + \frac{\mathrm{i}\omega\hat{\sigma}}{m} \boldsymbol{m} \wedge \boldsymbol{M}_1^1. \end{aligned} \tag{39}$$

$\boldsymbol{M}_3^1$ can be expressed as a function of $\boldsymbol{H}_3^1$ as in case $\tilde{\sigma} = 0$, so hence we get

$$L\boldsymbol{H}_3^1 = V_3^{1,0} + V_3^{1,\sigma} \tag{40}$$

where

$$L = \begin{pmatrix} \mathrm{i}\omega & 0 & \mu m_t \\ 0 & \mathrm{i}\gamma\omega & -\mu m_x \\ -(1+\alpha) m_t & \mu m_x & \mathrm{i}\gamma\omega \end{pmatrix}. \tag{41}$$

$V_3^{1,0}$ is the right-hand side of the analogue equation obtained in case $\tilde{\sigma} = 0$, and

$$V_3^{1,\sigma} = \frac{\mathrm{i}\omega\hat{\sigma}}{m} \boldsymbol{m} \wedge \boldsymbol{M}_1^1. \tag{42}$$

The solvability condition for equation (40) is obtained by applying the linear form

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \det \begin{pmatrix} \mathrm{i}\omega & 0 & x \\ 0 & \mathrm{i}\gamma\omega & y \\ -(1+\alpha) m_t & \mu m_x & z \end{pmatrix} \tag{43}$$

to the right-hand side and setting the result equal to zero.

One obtains the following result

$$\mathrm{i}A \frac{\partial g}{\partial \tau} + B \frac{\partial^2 g}{\partial \xi^2} + C g |g|^2 + \mathrm{i} D g = 0. \tag{44}$$

This is the nonlinear Schrödinger (NLS) equation

$$\mathrm{i}A \frac{\partial g}{\partial \tau} + B \frac{\partial^2 g}{\partial \xi^2} + C g |g|^2 = 0$$

already obtained in [4, 5], perturbed by the added term $\mathrm{i}Dg$. The real constants A, B, C have the same (complicated) expression as found in [4, 5], and

$$D = \frac{\hat{\sigma}}{m} \gamma^2 \omega^3 [2\mu m_x^2 + (\mu + (1+\alpha)\gamma) m_t^2]. \tag{45}$$

We verify that

$$\frac{D}{A} = \frac{1}{T_0} \tag{46}$$

where T_0 has expression (28).

If $p > 2$, i.e. if $\tilde{\sigma} \in o(\varepsilon^2)$, the term $\mathrm{i}Dg$ in equation (44) disappears and the evolution of the modulation is governed by the NLS equation

$$\mathrm{i}A \frac{\partial g}{\partial \tau} + B \frac{\partial^2 g}{\partial \xi^2} + C g |g|^2 = 0. \tag{47}$$

3. Examples of values of $\tilde{\sigma}$

The model we are dealing with has mainly been used for the description of the propagation of microwaves in ferrites. Experimental data about such materials can be found in various manuals [25, 27, 28]. Altough they are not up to date, we will restrict ourselves to the data given by these books. Values of $\tilde{\sigma}$ are computed from them using formulae (4) and (5). They vary from just less than 10^{-4} to nearly 10^{-1}.

The order of magnitude of $\tilde{\sigma}$ depends on the material, but we have to compare it with the small parameter ε, which seems to be arbitrary. In the present frame, ε represents both the ratio $\|\varepsilon \boldsymbol{H}_1^1\|/\|\boldsymbol{H}_0^0\|$ of the amplitude of the wave field to the exterior field, and the ratio $\xi/(x - Vt)$, the inverse of the ratio of the typical length of the amplitude variation to the one of the phase variation (the wavelength). These two quantities are assumed to have the same order of magnitude, depending on the experimental conditions.

If $\tilde{\sigma}$ is not very small, as an example for a polycrystalline ferrite $\tilde{\sigma} \simeq 10^{-1}$ (for example, $Ni_{0.8}Zn_{0.2}Fe_2O_4$, sphere of diameter 0.75 mm, frequency 9400 MHz: $\tilde{\sigma} \simeq 0.08$ ([25], p 323)), ε cannot be larger than 10^{-1}, thus $\tilde{\sigma}$ is of order ε, or larger. Exponential decay and linear behaviour is thus observed, in every case where this theory is applicable.

Now consider a relatively small value of $\tilde{\sigma}$, for example $\tilde{\sigma} \simeq 10^{-2}$ (for example, monocrystalline spinel ferrite, $Mn_{0.98}Fe_{1.86}O_4$, sphere of diameter 0.25 mm, frequency 9300 MHz: $\tilde{\sigma} \simeq 0.0067$, 0.0076, 0.010 on each of the three cristallographic axes ([25], p 186)). If we choose $\varepsilon \simeq 10^{-2}$, that is, if we choose a wave intensity of the order of magnitude of a hundredth of the exterior field, and look for the variations of the amplitude in wave packets of some hundreds of wavelengths, $\tilde{\sigma}$ has the same order of magnitude as ε. Thus we will observe exponential decay and linear behaviour. For larger wave intensity and shorter pulses, as ε approaches 10^{-1} (the robustness of the NLS model allows us to think that such a value of ε will not be too large, and that the perturbative calculus will still be valid) we will have $\tilde{\sigma}$ of order ε^2, and thus the amplitude modulation will be described by the perturbed NLS equation (44).

For very small values of $\tilde{\sigma}$, $\tilde{\sigma} \simeq 10^{-4}$ (for yttrium garnet ferrites, Lecraw *et al* found the value $\Delta H \simeq 0.6$ Oe [29], which leads to $\tilde{\sigma} \simeq 8 \times 10^{-5}$. This was the lowest value known at this time), the exponential decay and linear behaviour is observed for very small values of ε, $\varepsilon \simeq 10^{-4}$. For slightly larger values, $\varepsilon \simeq 10^{-2}$, the modulation is described by the perturbed NLS equation (44), and for high intensities and short pulses, with ε just less than 10^{-1}, $\tilde{\sigma}$ becomes of order ε^3, and is thus negligible, and the modulation obeys the NLS equation (47).

Recent experimental works have measured microwave envelope solitons [6, 7] or dark solitons [8] in yttrium iron garnet films. Although our theory does not, strictly speaking, apply to thin films, because we assumed that the media were inifinite, these experimental results are very close to our theoretical conclusions. The two regimes allowing solitons or dark-solitons that correspond to the Benjamin–Feir instability studied in [5] have been observed.

De Gasperis *et al* [6] measured a threshold power input P_{th} above which the attenuation falls. They showed that P_{th} is proportional to the inverse squared pulse length $1/t_p^2$. This corresponds to the NLS model: t_p has the same order of magnitude as the slow variables ξ and T, that is $1/\varepsilon$, while the power P of the wave is proportional to the square of the field described by the term $\boldsymbol{H}_1^1$ in our expansion, and is thus of order ε^2. Therefore, regarding the ε-dependency, P is proportional to $1/t_p^2$ when the formation of a soliton first occurs. A remarkable feature is that the absorption of the soliton is practically negligible, which is not the case for the solutions of the perturbed NLS equation (44) [30]. This is explained by our theory: the perturbed NLS equation (44) describes the behaviour of the wave only when the nonlinearity and dispersion exactly balances the damping. If damping dominates, which happens for a power input below some threshold (and the suited pulse length), the exponential decay kills the nonlinear effects. On the other hand, if the power input is large enough (and the pulse length accordingly small), so that nonlineary and dispersion dominate the damping, the modulation of the wave is governed by an unperturbed NLS equation. This justifies the power output increase observed by De Gasperis *et al*.

4. The relaxation time

The relaxation time T_0 defined by (28) is also given by the following expression:

$$T_0 = \frac{m}{-\hat{\sigma}} \frac{2[(\mu^2 m_x^2/u^2) + (1+\alpha\gamma^2)\gamma^2\omega^2]}{\gamma^2\omega^2[\gamma^2\omega^2 + \mu^2(m_x^2 + m_t^2)]}. \tag{48}$$

Because $\hat{\sigma}$ is a negative constant, we see that, as was expected, this quantity is always positive. For high frequencies, T_0 has the following limit:

$$T_0 \underset{\omega\longrightarrow\infty}{\sim} \frac{4}{-\hat{\sigma}m} \frac{1}{(1+\cos^2\theta)}. \tag{49}$$

We have put

$$\begin{cases} m_x = m\cos\theta \\ m_t = m\sin\theta. \end{cases} \tag{50}$$

θ is thus the angle between the propagation direction and the exterior field. For high frequencies, T_0 varies between $2/|\hat{\sigma}|m$ and $4/|\hat{\sigma}|m$, depending on θ: its order of magnitude does not change much.

The corresponding decay length is given by

$$X_0 = VT_0 = \frac{m}{-\hat{\sigma}} \frac{2(1+\alpha)(b+1)}{\gamma\omega^2 u[2\mu - (1-\gamma)(b+1)]} \tag{51}$$

or

$$X_0 = \frac{m}{-\hat{\sigma}} \frac{2(1+\alpha)}{\gamma\omega^2 u} \frac{\mu^2 m_x^2 + \gamma^2\omega^2}{2\mu m^2 - (1-\gamma)m_t^2}. \tag{52}$$

5. The perturbed NLS equation

Equation (44) describes the evolution of the envelope of a short pulse in a dispersive, weakly nonlinear medium, with weak damping. As we wrote in the introduction, it has already been derived in the same physical context [7, 12], but not in such a rigorous way as we have done here. It has also been obtained in other situations, as in nonlinear optics [13] or hydrodynamics [14]. We shall in this section summarize the known properties of this equation, and give a new exact solution for it.

Whatever the real non-zero values of the constants A, B, C, D, equation (44) is not integrable. For $D = 0$ it reduces to the integrable nonlinear Schrödinger (NLS) equation. It looks like the complex Ginzburg–Landau (CGL) [15–18] equation, which has the same form, but with B, C complex. Setting to zero the imaginary part of these parameters modifies totally the properties of the equation. Some particular case of the CGL equation, namely the case where the ratio B/C is real, is called the real Ginzburg–Landau equation. It has many properties in common with the so-called complex case. It is very important for our purpose not to mix up equation (44) with the real Ginzburg–Landau equation. Indeed, equation (44) is not a particular case of the CGL equation. It appears clearly when one considers the expressions of the exact solutions of CGL given in [15–17]. Divisions by the imaginary part of B and C occur in many places in these formulae. With the notation of these papers, $B = p = p_r + \mathrm{i}p_i$, and $C = q = q_r + \mathrm{i}q_i$, and the expressions are not defined as p_i or q_i are zero.

If, despite this observation, we try to retrieve in equation (44) the particular solutions of the CGL equations given by Nozaki and Bekki [15], we find a bilinear form, analogous to that of NLS [31]:

$$\begin{cases} [-\lambda + \mathrm{i}AD_\tau + BD_\xi^2]G \cdot F = 0 \\ \left[\dfrac{B}{C}D_\xi^2 + \dfrac{\mathrm{i}D - \lambda}{C}\right] F \cdot F = |G|^2. \end{cases} \tag{53}$$

D_X^n is the operator defined by Hirota as [31]

$$D_X^n f \cdot g = \left(\frac{\partial}{\partial_X} - \frac{\partial}{\partial_X'}\right)^n f(X)g(X')|_{X=X'}. \tag{54}$$

The functions F and G are such that F is real and $g = G/F$, and λ is an arbitrary complex constant. But any attempt to find an explicit solution for equation (44) generalizing one of the solutions of Nozaki and Bekki gives the constraint $D = 0$ as a solvability condition.

Karpman and Maslov [30] developed a perturbative method to find approximate solutions of this equation, when D is small. This method is based on the inverse scattering transform (IST) method for the resolution of the NLS equation. Thus it works only when this equation admits solitons, i.e. when $BC > 0$ (the sign of the BC product, in our special case, as a function of the physical parameters of the system, has been discussed in great detail in [5]). The quoted authors work with the rescaled equation:

$$\mathrm{i}u_{\tau'} + \tfrac{1}{2}u_{\xi'\xi'} + u|u|^2 = \mathrm{i}\rho u \tag{55}$$

(the subscript denotes partial differentiation). Equation (55) can be obtained from (44) through the transformation

$$\begin{cases} \xi' = a\xi \\ \tau' = \eta\tau \qquad \eta = \pm 1 \\ u = bg \end{cases} \tag{56}$$

with

$$\begin{cases} \eta = \mathrm{sgn} AB \\ a = \sqrt{\left|\dfrac{A}{2B}\right|} \\ b = \sqrt{\left|\dfrac{C}{A}\right|} \\ \rho = \dfrac{-\eta}{T_0}. \end{cases} \tag{57}$$

To first order in the perturbation theory, the solution is

$$u = 2\nu \frac{1}{\cosh z}\left[1 + \frac{\rho(4z^2 + \pi^2/3)}{16\mathrm{i}\nu^2}\right] \mathrm{e}^{\mathrm{i}(\delta + \mu_0 z/\nu)} \tag{58}$$

with

$$z = 2\nu(\xi' - 2\mu_0\tau' - \xi_0) \tag{59}$$

$$\nu = \nu_0\,\mathrm{e}^{2\rho\tau'} \tag{60}$$

$$\delta = \frac{\nu_0^2}{2\rho}[\mathrm{e}^{4\rho\tau'} - 1] + 2\mu_0^2\tau' + 2\mu_0\xi_0 + \delta_0. \tag{61}$$

μ_0, ν_0, δ_0, ξ_0 are arbitrary constants.

Neglecting the term proportional to ρ in the parenthesis in expression (58), we have the so-called adiabatic approximation: a solitonic-shaped solitary wave, where amplitude $\nu = \nu_0\,\mathrm{e}^{2\rho\tau'}$ is exponentially decreasing with time τ ($\rho < 0$ if $\eta = +1$, and $\rho > 0$ if $\eta = -1$), while its width $1/\nu$ is exponentially increasing (with time τ).

The correction term $\rho(4z^2 + \pi^2/3)/16\mathrm{i}\nu^2$ is proportional to

$$\frac{\rho\pi^2}{48\mathrm{i}\nu_0^2}\mathrm{e}^{-4\rho\tau'}$$

when t is large ($\tau > 0$), and thus increases very rapidly. The perturbation calculus is therefore valid only for small τ',

$$|\tau'| < \frac{1}{|\rho|}. \tag{62}$$

This restriction is not surprising: once we return to the laboratory frame, the rescaled value of ρ and is proportional to $\tilde{\sigma}$, thus is of order ε^2. Therefore, condition (62) means that solution (58) is valid only for times t of order ε^{-2}, but the scaling (10) defines $\tau(= \pm\tau') = \varepsilon^2 t$, which means precisely that we restrict ourselves to values of the time t of order ε^{-2}. This upper limit for t has in fact a sufficiently large value.

Numerical studies of the solution have also been performed, in relation to the applications to nonlinear optics. [32] presents such a calculation, and compares its results to those of the previous perturbative calculus. The authors retrieve the validity condition (62) of the perturbative calculus. For larger values of time t (replaced by the spatial parameter z in their study), the behaviour of the numerical solution is found to be identical to that of the linear solution. An analogous feature can be found in the frame of our present work by the following method. Let us call $t = 1/\varepsilon^2$ the time scale. Then, considering values of the time variable t greater than $1/\rho$ is equivalent to assuming that the damping constant ρ has an order of magnitude greater than ε^2. We have seen that, in this latter case, the behaviour of the wave was linear. Authors of [32] have made the same observation, although their reasoning concerns equation (44), and not on the basic set.

In addition to these already known properties of equation (44), we will describe a method that gives an exact particular solution of the equation, and a way to compute various approximate solutions. Let

$$p = \frac{B}{A} \qquad q = \frac{C}{A} \qquad r = \frac{D}{A}. \tag{63}$$

Equation (44) reads

$$\mathrm{i}g_\tau + pg_{\xi\xi} + qg|g|^2 + \mathrm{i}rg = 0. \tag{64}$$

We put

$$g(\xi,\tau) = h(\xi,\tau)\,\mathrm{e}^{-r\tau}. \tag{65}$$

Then the equation verified by function h is

$$\mathrm{i}h_\tau + ph_{\xi\xi} = -qh|h|^2\,\mathrm{e}^{-2r\tau}. \tag{66}$$

When τ is large in regard to $T_0 = 1/r$, the right-hand side of equation (66) can be treated as a perturbation. Although we just write that τ normally has the same order of magnitude as T_0, the exponential $\mathrm{e}^{-2r\tau}$ decreases very rapidly, and will be negligible for values of τ not too far from T_0. Let us expand h in a power series of the quantity $\mathrm{e}^{-2r\tau}$:

$$h = \sum_{n=0}^{+\infty} h^{(n)} \mathrm{e}^{-2nr\tau}. \tag{67}$$

Each function $h^{(n)}$ should verify

$$\mathrm{i}(h_\tau^{(n)} - 2nrh^{(n)}) + ph_{\xi\xi}^{(n)} = \Phi^{(n)} \tag{68}$$

$$\Phi^{(n)} = -q \sum_{\substack{n_1+n_2+n_3+1=n \\ n_1, n_2, n_3 \geqslant 0}} h^{(n_1)} h^{(n_2)} h^{(n_3)*}. \tag{69}$$

For each n, $\Phi^{(n)}$ is defined by the knowledge of $h^{(r)}$, $0 \leqslant r \leqslant n-1$, and thus equations (68) and (69) constitute a recurrence relation for the coefficients $h^{(n)}$ of series (67).

For $n = 0$, we get

$$\mathrm{i}h_\tau^{(0)} + ph_{\xi\xi}^{(0)} = 0. \tag{70}$$

Let us first choose the particular solution

$$h^{(0)} = \mathcal{A}\, \mathrm{e}^{\mathrm{i}(K\xi - pK^2\tau)} \tag{71}$$

where $\mathcal{A} \in \mathbb{C}$, $K \in \mathbb{R}$ are arbitrary constants. Starting with this solution for $h^{(0)}$, we will construct an explicit exact solution of equation (66), using the following method.

First, we show by induction that a solution of the system formed by equations (68) and (69) for each n is such that, for each $n \geqslant 1$,

$$h^{(n)} = \lambda_n h^{(0)} \tag{72}$$

where λ_n is a constant.

Then, we show, always by induction, that $\lambda_n = u_n \lambda_1^n$, with $\lambda_1 = (-iq/2r)|\mathcal{A}|^2$, and u_n is a sequence of real constants defined by

$$\begin{cases} u_0 = 1 \\ u_n = \dfrac{1}{n} \displaystyle\sum_{n_1+n_2+n_3+1=n} (-1)^n u_{n_1} u_{n_2} u_{n_3}. \end{cases} \tag{73}$$

Expanding $\mathrm{e}^X = \mathrm{e}^X \cdot \mathrm{e}^X \cdot \mathrm{e}^{-X}$, we show that $u_n = 1/n!$ for each n. It is then easy to sum up the series. Finally we have

$$g = \mathcal{A} \exp\left[\mathrm{i}(K\xi - pK^2\tau) - r\tau - \frac{\mathrm{i}q}{2r}|\mathcal{A}|^2\, \mathrm{e}^{-2r\tau}\right]. \tag{74}$$

Starting with

$$h^{(0)} = \mathcal{A}_1\, \mathrm{e}^{\mathrm{i}\psi_1} + \mathcal{A}_2\, \mathrm{e}^{\mathrm{i}\psi_2} \tag{75}$$

$$\mathcal{A}_1, \mathcal{A}_2 \in \mathbb{C} \qquad \psi_j = K_j\xi - pK_j^2\tau \qquad j = 1,\ 2$$

we obtain a ‘two-modes’ solution, that cannot be exactly computed, but for which we can give asymptotic expansion for large values of τ. We report expression (75) of $h^{(0)}$ in the equation for $h^{(1)}$:

$$\mathrm{i}(h_\tau^{(1)} - 2rh^{(1)}) + ph_{\xi\xi}^{(1)} = -qh^{(0)}|h^{(0)}|^2. \tag{76}$$

We expand the right-hand side and seek for an analogous expression for $h^{(1)}$.

After a short computation, we get the following approximate solution of (44),

$$g = \mathrm{e}^{-r\tau}[\mathcal{A}_1\,\mathrm{e}^{\mathrm{i}\psi_1} + \mathcal{A}_2\,\mathrm{e}^{\mathrm{i}\psi_2} + \mathrm{e}^{-2r\tau}(\lambda_1^{(1)}\mathrm{e}^{\mathrm{i}\psi_1} + \lambda_1^{(2)}\mathrm{e}^{\mathrm{i}\psi_2} + \lambda_1^{(21)}\mathrm{e}^{2\mathrm{i}\psi_2-\mathrm{i}\psi_1} + \lambda_1^{(12)}\mathrm{e}^{2\mathrm{i}\psi_1-\mathrm{i}\psi_2}) + \mathrm{O}(\mathrm{e}^{-rt})] \tag{77}$$

with:

$$\lambda_1^{(1)} = \frac{q}{2\mathrm{i}r}\mathcal{A}_1[|\mathcal{A}_1|^2 + 2|\mathcal{A}_2|^2] \tag{78}$$

$$\lambda_1^{(2)} = \frac{q}{2\mathrm{i}r}\mathcal{A}_2[|\mathcal{A}_2|^2 + 2|\mathcal{A}_1|^2] \tag{79}$$

$$\lambda_1^{(21)} = \frac{q\mathcal{A}_1^*\mathcal{A}_2^2}{2(\mathrm{i}r + p(K_1 - K_2)^2)} \tag{80}$$

$$\lambda_1^{(12)} = \frac{q\mathcal{A}_2^*\mathcal{A}_1^2}{2(\mathrm{i}r + p(K_1 - K_2)^2)}. \tag{81}$$

6. Conclusion

We have discussed the effect of the damping on the nonlinear modulation of a monochromatic wave in a ferromagnet. Its behaviour depends on the magnitude of the damping constant. If this constant is small enough, in relation to the intensity of the wave, the evolution of the modulation is governed by the NLS equation, and the formation of solitons cancels the effects of the damping. If the damping constant is larger, the wave decreases exponentially, and no nonlinear modulation is observed. The multiscale expansion has been pursued to the following order: the evolution equation is still linear.

It is also possible that damping exactly balances the nonlinearity and dispersion; in this case the evolution of the modulation is described by a perturbed NLS equation. We have summarized the known properties of this equation and given a new exact solution for it, and a method to compute approximate particular solutions.

Some questions about the physical relevance of the present model have been pointed out; in particular, we have shown that the inhomogeneous exchange interaction does not modify the nonlinear behaviour of the wave. We have also commented on our conclusions in relation to standard numerical data, and re-interpreted the experiment of De Gasperis *et al*.

Appendix A

In this appendix, we show that, in the frame where we derived the perturbed NLS equation (44), a small inhomogeneous exchange interaction term has no effect on the modulation of the monochromatic wave, but introduces only a linear phase modulation. The precise expressions for the coefficients of this modulation are very complicated and not essential, and thus in order to avoid numerous pages of formulae we shall omit them, as well as many technical details of the calculus.

The inhomogeneous exchange interaction can be described [12], in the frame of the present model, by replacing equation (3) by

$$\frac{\partial \boldsymbol{M}}{\partial \boldsymbol{t}} = -\delta\mu_0 \boldsymbol{M} \wedge \boldsymbol{H}_{\mathrm{eff}} + \frac{\sigma}{\|\boldsymbol{M}\|}\boldsymbol{M} \wedge (\boldsymbol{M} \wedge \boldsymbol{H}) \tag{82}$$

where the field H is replaced by the effective field:

$$\boldsymbol{H}_{\mathrm{eff}} = \boldsymbol{H} + \rho\Delta\boldsymbol{M}. \tag{83}$$

ρ is some real constant. As previously given, we rescale the equations, and then use the expansion defined by equations (8)–(12). We assume that $\tilde{\sigma}$ and ρ are small, in the following sense,

$$\tilde{\sigma} = \varepsilon^2 \hat{\sigma} \qquad \text{and} \qquad \rho = \varepsilon \hat{\rho} \tag{84}$$

where $\hat{\rho}$, $\hat{\sigma}$ are quantities of order zero. Physically, it would mean that the coefficient σ is very small in relation to ρ, which is not the case in general, but we intend here to prove, in a mathematical way, that the influence of the inhomogeneous exchange interaction on the nonlinear behaviour is negligible, even when the coefficient ρ is not so small. That is why we assume such an order of magnitude for ρ.

The terms at order ε^0 are the same as previously given by equations (14)–(16). At order ε^1, we also find the solution given by equations (17)–(23). Because we have assumed that ρ is small, of order ε, the inhomogeneous exchange interaction does not modify the dispersion relation (this means that we are not considering spin waves). At order ε^2, we obtain an equation analogous, in some sense, to equation (24):

$$\frac{\partial g}{\partial T} + V\frac{\partial g}{\partial \xi} - \mathrm{i}\Omega g = 0. \tag{85}$$

The real coefficient Ω reads

$$\Omega = \hat{\rho}k^2 \frac{\gamma\omega[2\mu - (1+b)(1-\gamma)]}{2(1-\gamma)(1+\alpha)(b+1+\gamma\mu u^2)}. \tag{86}$$

Thus

$$g(\xi, T, \tau) = h(\xi - VT, \tau)\, \mathrm{e}^{\mathrm{i}\Omega T}. \tag{87}$$

Because of the factor i in equation (85), the exponential phase factor will not kill the nonlinear effects as did the exponential decaying factor in equation (29). For this reason, we will obtain a nonlinear evolution equation with ρ as large as order ε, while such a result necessitates σ as small as order ε^2.

The next step of the calculus reads as in the case where we neglected the inhomogeneous exchange term. The expressions obtained for $\boldsymbol{H}_2^0$, $\boldsymbol{M}_2^0$ are the same as in [4, 5]. The terms $\boldsymbol{H}_2^1$ and $\boldsymbol{M}_2^1$ contain a term proportional to $\hat{\rho}$, which we will not write here. At order ε^3, we have to write a solvability condition. A term proportional to $\hat{\rho}$ appears in equation (82); this is the coefficient of $\varepsilon^3 \mathrm{e}^{\mathrm{i}\varphi}$ in the expansion of

$$\rho \boldsymbol{M} \wedge (\Delta \boldsymbol{M}). \tag{88}$$

After use of equation (87), we obtain the equation

$$\mathrm{i}Ah_\tau + Ph + \mathrm{i}Rh_\xi + Bh_{\xi\xi} + \mathrm{i}Dh + Ch|h|^2 = 0. \tag{89}$$

The important fact is that the coefficients A, B, C, D have the same value as in the case where $\rho = 0$. The coefficients P and R have complicated expressions, which we shall omit, with P proportional to $\hat{\rho}^2$ and R porportional to $\hat{\rho}$. Let

$$h(\xi - VT, \tau) = j(\xi - VT, \tau)\, \mathrm{e}^{\mathrm{i}(a(\xi - VT) + b\tau)} \tag{90}$$

with

$$a = \frac{-R}{2B} \tag{91}$$

and

$$b = \frac{1}{A}[-a^2 B - aR + P]. \tag{92}$$

Then equation (89) reduces to

$$\mathrm{i}Aj_\tau + Bj_{\xi\xi} + Cj|j|^2 + \mathrm{i}Dj = 0 \tag{93}$$

which is identical to equation (44). Thus the only effect of the inhomogeneous exchange term is the linear phase factor

$$\exp \mathrm{i}(\Omega T + a(\xi - VT) + b\tau)$$

which gives account of the modification of the linear dispersion relation by the inhomogeneous exchange interaction. However, the nonlinear behaviour of the modulation is not affected.

Appendix B

In this appendix we give the lengthy formulae that permit us to compute the coefficients E and F in equation (36). First, we compute the second-order terms:

$$\boldsymbol{H}_2^1 = \boldsymbol{h}_1^1 + \boldsymbol{h}_2^{1,\xi} g_\xi + \boldsymbol{h}_2^{1,\sigma} g \tag{94}$$

$$\boldsymbol{M}_2^1 = \boldsymbol{m}_1^1 + \boldsymbol{m}_2^{1,\xi} g_\xi + \boldsymbol{m}_2^{1,\sigma} g. \tag{95}$$

The function r that intervenes in equation (30) is such that

$$f(\xi, T, \tau) = r(\xi - VT, T, \tau)\,\mathrm{e}^{-T/T_0}. \tag{96}$$

$\boldsymbol{m}_2^{1,\xi}$, $\boldsymbol{h}_2^{1,\xi}$, $\boldsymbol{m}_2^{1,\sigma}$, $\boldsymbol{h}_2^{1,\sigma}$ are constant vectors given by

$$\boldsymbol{m}_2^{1,\xi} = \frac{1}{u^2}(V-u)\begin{pmatrix} -m_t(b+1+2\alpha\gamma) \\ m_x(b+1+2\alpha\gamma) \\ 2\mathrm{i}\gamma\omega \end{pmatrix} \tag{97}$$

$$\boldsymbol{h}_2^{1,\xi} = \frac{1}{u^2}(V-u)\begin{pmatrix} m_t(b+1+2\alpha\gamma) \\ (m_x/\gamma)(1-b) \\ 0 \end{pmatrix} \tag{98}$$

$$\boldsymbol{m}_2^{1,\sigma} = \begin{pmatrix} (-m_t/\omega)[(\gamma/T_0)[2\alpha(1-\gamma)-\mu] - (\gamma^2\omega^2\tilde{\sigma}/m)] \\ (m_x/\omega)[(\gamma/T_0)[2\alpha(1-\gamma)-\mu] - (\gamma^2\omega^2\tilde{\sigma}/m)] \\ 2\mathrm{i}\gamma(1-\gamma)/T_0 \end{pmatrix} \tag{99}$$

$$\boldsymbol{h}_2^{1,\sigma} = \begin{pmatrix} (m_t/\omega)[(\gamma/T_0)[2\alpha(1-\gamma)-\mu] - (\gamma^2\omega^2\tilde{\sigma}/m)] \\ (m_x/\gamma\omega)[(1/T_0)[2(1-\gamma)+\gamma\mu] + (\gamma^2\omega^2\tilde{\sigma}/m)] \\ 0 \end{pmatrix}. \tag{100}$$

Then we write the solubility condition for the system (6) and (7) at order ε^3 for the fundamental frequency. Thus leads to equation (30), after we have replaced g using equation (29). The coefficients read

$$E = \mathcal{D}\boldsymbol{V}_3^{1,(1)} + \mathcal{D}_1\boldsymbol{U}_3^{1,(1)} \tag{101}$$

$$F = \mathcal{D}\boldsymbol{V}_3^{1,(0)} + \mathcal{D}_1\boldsymbol{U}_3^{1,(0)}. \tag{102}$$

The linear forms $\mathcal{D}$ and $\mathcal{D}_1$ are given by

$$\mathcal{D} = (\gamma(1+\alpha)m_t, -\mu m_x, \mathrm{i}\gamma\omega) \tag{103}$$

$$\mathcal{D}_1 = \frac{\omega}{\mu}(\mathrm{i}\gamma\mu m_t, -\mathrm{i}\mu m_x, -\gamma\omega). \tag{104}$$

$\boldsymbol{U}_3^{1,(0)}$, $\boldsymbol{U}_3^{1,(1)}$, $\boldsymbol{V}_3^{1,(0)}$, $\boldsymbol{V}_3^{1,(1)}$ are vector coefficients of the linear system whose compatibility condition is equation (30). They read

$$U_3^{1,(0),x} = \frac{2\mathrm{i}}{\omega T_0}(h_2^{1,\sigma,x} + m_2^{1,\sigma,x}) \tag{105}$$

$$U_3^{1,(0),s} = \frac{2\mathrm{i}}{\omega T_0}(h_2^{1,\sigma,s} + m_2^{1,\sigma,s}) + \frac{1-\gamma}{\omega^2 T_0} h_1^{1,s} \qquad \text{for } s = y, z \tag{106}$$

$$U_3^{1,(1),x} = \frac{2\mathrm{i}}{\omega}\left(\frac{1}{T_0}(h_2^{1,\xi,x} + m_2^{1,\xi,x}) + V(h_2^{1,\sigma,x} + m_2^{1,\sigma,x})\right) \tag{107}$$

$$U_3^{1,(1),s} = \frac{2\mathrm{i}}{\omega}\left(\frac{1}{T_0}(h_2^{1,\xi,s} + m_2^{1,\xi,s}) + V(h_2^{1,\sigma,s} + m_2^{1,\sigma,s})\right) + \frac{2V(1-\gamma)}{\omega^2 T_0} h_1^{1,s} + \frac{2\mathrm{i}k}{\omega} h_2^{1,\sigma,s} \qquad \text{for } s = y, z \tag{108}$$

$$\boldsymbol{V}_3^{1,(0)} = \frac{1}{T_0}\boldsymbol{m}_2^{1,\sigma} + \frac{\sigma}{m}\boldsymbol{m} \wedge [\boldsymbol{m} \wedge (\boldsymbol{h}_2^{1,\sigma} - \alpha \boldsymbol{m}_2^{1,\sigma})] \tag{109}$$

$$\boldsymbol{V}_3^{1,(1)} = \frac{1}{T_0}\boldsymbol{m}_2^{1,\xi} + V\boldsymbol{m}_2^{1,\sigma} + \frac{\sigma}{m}\boldsymbol{m} \wedge [\boldsymbol{m} \wedge (\boldsymbol{h}_2^{1,\xi} - \alpha \boldsymbol{m}_2^{1,\xi})]. \tag{110}$$

These quantities can be computed explicitly using equations (97)–(100); then the result can be found from equations (101) and (102). This gives the coefficients E and F.

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